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Displaced Diffusion Option Pricing

MARK RUBINSTEIN*

ABSTRACT

This paper develops a new option pricing formula that pushes the underlying source of risk back to the risk of individual assets of the firm. The formula simultaneously encompasses differential riskiness of the assets of the firm, their relative weights in determining the value of the firm, the effects of firm debt, and the effects of a dividend policy with both constant and random components. Although this setting considerably generalizes the Black-Scholes [1] analysis, it nonetheless produces a formula via riskless arbitrage arguments that, given estimated inputs, is as easy to use as the Black-Scholes formula.

I. The Basic Model

SUPPOSE WE CONCEIVE OF a firm in the following way. It only holds two assets, one of which is risky and the other riskless. The value of the riskless asset follows a lognormal diffusion process with an annualized instantaneous volatility σ_R and the riskless asset grows at the discrete annualized compounded rate of $r - 1$. At present, $0 \leq \alpha \leq 1$ proportion of the total value V of the firm is invested in the risky asset and $(1 - \alpha)$ in the riskless asset. Then, at the end of time t years, the value of the firm would be

$$[\alpha e^y + (1 - \alpha)r^t]V$$

where y is a normally distributed random variable with instantaneous volatility $\sigma_R \sqrt{t}$.

In addition, suppose the firm has a very simple capital structure with a debt-equity ratio of β measured in terms of the current market values of its stock and bonds. If S is the current market value of the stock, then if there were no dividends and the debt were riskless, the value of the stock, S^* , after time t would be

$$S^* = [\alpha e^y + (1 - \alpha)r^t](1 + \beta)S - \beta S r^t$$

which can be rewritten

$$S^* = [\alpha(1 + \beta)e^y + (1 - \alpha - \alpha\beta)r^t]S$$

Incidentally, to insure the debt is riskless, we must require $1 - \alpha - \alpha\beta \geq 0$ so that

$$\beta \leq \frac{1 - \alpha}{\alpha}.$$

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More generally, suppose that this firm has the following dividend policy. At each ex-dividend date, $k \leq t$ time in the future, the firm will pay at least and for certain an amount $d_k S$. To be certain, this is paid out of the riskless portion of the value of the stock $(1 - \alpha - \alpha\beta)r^k S$. Therefore

$$\sum_{k \leq t} d_k r^{-k} \leq 1 - \alpha - \alpha\beta$$

In addition, at each ex-dividend date, k time in the future, the firm pays out to its shareholders a deterministic fraction δ_k of the value at that time of its risky asset. Therefore, after time k , the total dividend at that time is

$$D_k = [\alpha(1 + \beta)(\prod_{\tau < k} (1 - \delta_\tau))\delta_k e^y + d_k]S$$

All of this implies that the stock value after time t will be divided into two components: (1) a risky portion

$$[\alpha(1 + \beta)(\prod_{k \leq t} (1 - \delta_k))e^y]S \equiv ae^y S$$

and (2) a riskless portion

$$[(1 - \alpha - \alpha\beta)r^t - \sum_{k \leq t} d_k r^{t-k}]S \equiv bS.$$

With only one uncertain state variable y moving this whole structure, we know from option pricing theory that provided this state variable is "tradable" and we permit continuous trading with a known rate of interest, we can value an option on this stock with riskless arbitrage arguments. In this case, the state variable is tradable indirectly by forming a portfolio of the firm's stock and riskless bonds. By the Cox-Ross [2] reasoning, then the current price of a European call option on the stock is just the present value of its expected future value under risk neutrality. In particular, for a call with striking price K and time to expiration t , its expected future value is

$$E[ae^y S + bS - K | ae^y S + bS \geq K]$$

This expectation is formally equivalent to the classical Black-Scholes case, except, in that case, it is necessary to take the expectation

$$E[e^z S - K | e^z S \geq K]$$

where e^z is the compounded return of the stock and z follows a lognormal diffusion process with instantaneous volatility σ . Therefore, the solution to our more general problem will be the Black-Scholes formula $C(S, K, t, r, \sigma)$ except where the call is evaluated at

$$C(aS, K - bS, t, r, \sigma_R).$$

Writing this all out

$$C = aSN(x) - (K - bS)r^{-t}N(x - \sigma_R \sqrt{t})$$

where

$$x \equiv \frac{\log(aS/(K - bS)r^{-t})}{\sigma_R \sqrt{t}} + \frac{1}{2} \sigma_R \sqrt{t}$$

$$a \equiv \alpha(1 + \beta) \prod_{k \leq t} (1 - \delta_k)$$

$$b \equiv (1 - \alpha - \alpha\beta)r^t - \sum_{k \leq t} d_k r^{t-k}$$

II. Discussion

Relative to the Black-Scholes formula, this displaced diffusion formula has several desirable features.

First, it contains the Black-Scholes formula as a special case, $a = b = 0$. Second, it derives the stochastic process driving the stock from a more fundamental analysis of the characteristics of the firm. Like the compound option model (Geske [3]), it brings in debt but it goes deeper into the structure of the firm by decomposing its assets. Since its assets consist of a portfolio of a risky and riskless asset, the volatility of the value of the firm will not be a constant as in the compound option case, but rather it will be stochastic. For example, if the value of the firm rises very quickly it will be primarily because of the contribution of the risky component. This will shift the portfolio composition toward the risky component, and, since its own volatility is assumed constant, this will end up increasing the volatility of the value of the firm. On the other hand, if the value of the firm falls or rises more slowly than r , then the volatility of the value of the firm will fall.

To be sure, no firm holds only two assets, one of which is riskless. This simplification, so helpful in modeling, is really an approximation to reality for firms that have two types of assets: those that are *relatively* risky and those that are *relatively* riskless. The former might be future investment opportunities and the latter, assets in place from past investments. Or, as David Pyle has suggested to me, a single asset of the firm can be decomposed in principle into a portion providing a riskless return and a portion providing a risky return.

Furthermore, the assumption of more than one risky asset poses several problems. First, it requires a more complex arbitrage argument and both portions of the firm need to be separately tradable. Second, even assuming risk neutrality, the solution for the option value seems to require numerical integration. Third, it increases the number of parameters that must be estimated.

If there were no debt ($\beta = 0$), then the stock volatility would tend to rise with large and quick increases in the stock price. However, debt (as in the compound option model) works in just the opposite way. Whether the volatility of the stock will rise or fall with increases in the stock will depend jointly on the influences of the asset composition, α , and the debt-equity ratio, β . Although the debt is not risky, the assumption of riskless debt is reasonable for most firms with listed options, since preliminary analysis of the compound option model shows that the levels of debt risk usually present are sufficiently low to have negligible impact on option values.

Table I
Representative Displaced Diffusion Call Values

$\beta = 0$ $\delta_k = d_k = 0$
 $S = 40$ $r = 1.05$

σ	K	$\alpha = 1$ (Black-Scholes)						$\alpha = .75$						$\alpha = .5$					
		Expiration Month ^a																	
		January	April	July	January	April	July	January	April	July	January	April	July	January	April	July	January	April	July
.2	30	10.12	10.49	10.88	10.12	10.49	10.87	10.12	10.49	10.87	10.12	10.48	10.85	10.12	10.48	10.85	10.12	10.48	10.85
	35	5.15	5.76	6.40	5.15	5.74	6.37	5.15	5.74	6.37	5.15	5.71	6.31	5.15	5.71	6.31	5.15	5.71	6.31
	40	1.00	2.17	3.00	1.00	2.16	3.01	1.00	2.16	3.01	1.00	2.17	3.00	1.00	2.17	3.00	1.00	2.17	3.00
	45	.02	.51	1.10	.03	.53	1.15	.03	.53	1.15	.03	.58	1.23	.03	.58	1.23	.03	.58	1.23
	50	—	.08	.32	—	.09	.37	—	.09	.37	—	.12	.46	—	.12	.46	—	.12	.46
.3	30	10.12	10.58	11.14	10.12	10.56	11.09	10.12	10.56	11.09	10.12	10.52	10.98	10.12	10.52	10.98	10.12	10.52	10.98
	35	5.22	6.25	7.17	5.21	6.22	7.13	5.20	6.22	7.13	5.20	6.14	7.02	5.20	6.14	7.02	5.20	6.14	7.02
	40	1.46	3.07	4.19	1.46	3.09	4.22	1.46	3.09	4.22	1.46	3.10	4.25	1.46	3.10	4.25	1.46	3.10	4.25
	45	.16	1.26	2.24	.17	1.32	2.33	.20	1.42	2.51	.20	1.42	2.51	.20	1.42	2.51	.20	1.42	2.51
	50	.01	.44	1.11	.01	.50	1.23	.02	.62	1.48	.02	.62	1.48	.02	.62	1.48	.02	.62	1.48
.4	30	10.13	10.83	11.65	10.13	10.78	11.56	10.12	10.78	11.56	10.12	10.66	11.34	10.12	10.66	11.34	10.12	10.66	11.34
	35	5.39	6.89	8.10	5.37	6.86	8.07	5.33	6.86	8.07	5.33	6.76	7.98	5.33	6.76	7.98	5.33	6.76	7.98
	40	1.92	3.98	5.37	1.92	4.02	5.46	1.93	4.02	5.46	1.93	4.07	5.61	1.93	4.07	5.61	1.93	4.07	5.61
	45	.42	2.10	3.43	.44	2.21	3.62	.50	2.40	3.97	.50	2.40	3.97	.50	2.40	3.97	.50	2.40	3.97
	50	.06	1.03	2.12	.07	1.16	2.37	.10	1.41	2.84	.10	1.41	2.84	.10	1.41	2.84	.10	1.41	2.84

NOTE: The parameter σ_R is set so that the $\text{Var}[\log(S^*/S)]$ is identical irrespective of α . That is,

$$\sigma^2 t = \int_{-\infty}^{\infty} [\log(\alpha e^y + (1 - \alpha)r')]^2 f(y) dy - \left[\int_{-\infty}^{\infty} \log(\alpha e^y + (1 - \alpha)r') f(y) dy \right]^2$$

where

$$f(y) = \frac{1}{\sigma_R \sqrt{2\pi t}} \exp[-(y - \mu_R t)^2 / 2\sigma_R^2 t] \quad \text{and} \quad \mu_R t = t \log r - \frac{1}{2} \sigma_R^2 t$$

^a The January options have one month to expiration; the Aprils four months; and the Julys seven months. r and σ are expressed in annualized units.

Fourth, unlike the classical Black-Scholes formula which only readily admits a deterministic dividend yield, the displaced diffusion model also admits a constant component to dividend payouts. It seems probable that this is a more realistic model of dividend policy.

For the simple case of no debt ($\beta = 0$) and no dividends ($\delta_k = d_k = 0$), Table I compares Black-Scholes and displaced diffusion European call values. Observe that compared to the Black-Scholes values, out-of-the-money calls tend to be worth more, in-the-moneys worth less, and these differences become larger in relative terms as the expiration date approaches. Very rough casual empiricism suggests that the displaced diffusion model appears to move option values in the right directions, relative to the Black-Scholes formula.

However, by comparison with the Black-Scholes formula, this improvement is not without cost: measurement of the risk characteristics of the stock is more complex. Even if $\beta = 0$ and $\delta_k = d_k = 0$, it is still necessary to estimate α and σ_R . One possibility is to use accounting information. For example, the ratio of book to market value may give a fair estimate of $1 - \alpha$. A more promising approach would be to infer α and σ_R from past stock price movements. Although α changes over time, it is helpful to observe that once α is determined for any single date in the past, knowing the path of the stock price from that point forward would allow one to determine the corresponding path of α exactly.

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